# MESA vs Goertzel-DFT



MESA which stands for Maximum Entropy Spectral Analysis is a widely used mathematical technique designed to find the frequencies present in data. MESA was developed by J.P Burg for his Ph.D dissertation at Stanford University in 1975. The use of the MESA technique for stocks has been written about in many articles and has been popularized as a trading technique by John Ehlers.

The Fourier Transform is a mathematical technique named after the famed French mathematician Jean Baptiste Joseph Fourier 1768-1830. It's digital form, namely the discrete-time Fourier Transform (DFT) series, is a widely used mathematical technique to find the frequencies of discrete time sampled data. The use of the DFT has been written about in many articles in this magazine (see references section).

Today, both MESA and DFT are widely used in science and engineering in digital signal processing. The application of MESA and Fourier mathematical techniques are prevalent in our everyday life from everything from television to cell phones to wireless internet to satellite communications.

## MESA Advantages & Disadvantage

MESA is a mathematical technique that calculates the frequencies of a time series from the autoregressive coefficients of the time series. We have all heard of regression. The simplest regression is the straight line regression of price against time where **price(t) = a+b*t** and where **a** and **b** are calculated such that the square of the distance between price and the best fit straight line is minimized (also called least squares fitting). With autoregression we attempt to predict tomorrows price by a linear combination of **M** past prices in the formula:

$$p_{est}(n) = -\sum_{k=1}^{M} a[k]*p[n-k]$$

where **a[1]…a[M]** are the autoregression coefficients, $\mathbf{p_{est}(n)}$ is the least squares estimate of **p(n)** and **p(n-k)** is the price at the sampling period (n-k) bars ago. If we have **N** prices where **N** is much greater than **M**, it turns out that if we solve for the coefficients **a[1]…a[M]** by minimizing

$$error = \sum_{n=M+1}^{N} [p(n)-p_{est}(n)]^2 \qquad (1)$$

then the M coefficients solved for can be used in formula (2) to determine the power of any frequency f. P(f) will give a peak in amplitude at f is that frequency exists.

$$P(f) = xms / [\ \{[1+\sum_{k=1}^{M} a[k]*\cos(2*\pi*f*k)\}^2 + \{\sum_{k=1}^{M} a[k]*\sin(2*\pi*f*k)\}^2\ ] \qquad (2)$$

where xms is the mean square error (Equation 1) and f is the frequency of interest. The C++ code to solve for the above can be found in "Numerical Recipes in C++", by Press et al (see reference section)

One of the major advantages of MESA is that the frequency examined is not constrained to multiples of 1/N (1/N is equal to the DFT frequency spacing and N is equal to the number of sample points). For instance with the DFT and N data points we can only look a frequencies of 1/N, 2/N, ….., 0.5. With MESA we can examine any frequency band within that range and any frequency spacing between i/N and (i+1)/N . For example, if we had 100 bars of price data, we might be interested in looking for all cycles between 3 bars per cycle and 30 bars/ cycle only and with a frequency spacing of 0.5 bars/cycle. DFT would examine all bars per cycle of between 2 and 50 with a frequency spacing constrained to 1/100.

Another of the major advantages of MESA is that the dominant spectral (frequency) peaks of the price series, if they exist, can be identified with fewer samples than the DFT technique. For instance if we had a 10 bar price period and a high signal to noise ratio we could accurately identify this period with 40 data samples using the MESA technique. This same resolution might take 128 samples for the DFT. One major ***disadvantage*** of the MESA technique is that with low signal to noise ratios, that is below 6db (signal amplitude/noise amplitude < 2), the ability of MESA to find the dominant frequency peaks is severely diminished.(see Kay, Ref 10, p 437). With noisy price series this disadvantage can become a real problem. Another disadvantage of MESA is that when the dominant frequencies are found another procedure has to be used to get the amplitude and phases of these found frequencies. This two stage process can make MESA much slower than the DFT and FFT . The FFT stands for Fast Fourier Transform. The Fast Fourier Transform(FFT) is a computationally efficient algorithm which is a designed to rapidly evaluate the DFT. We will show in examples below the comparisons between the DFT & MESA using constructed signals with various noise levels.

## DFT Advantages and Disadvantages.

The mathematical technique called the DFT takes a discrete time series(price) of N equally spaced samples and transforms or converts this time series through a mathematical operation into set of N complex numbers defined in what is called the frequency domain. Why would we what to do that? Well it turns out that we can do all kinds of neat analysis tricks in the frequency domain which are just to hard to do, computationally wise, with the original price series in the time domain. If we make the assumption that the price series we are examining is made up of signals of various frequencies plus noise, than in the frequency domain we can easily filter out the frequencies we have no interest in and minimize the noise in the data. We could then transform the resultant back into the time domain and produce a filtered price series that hopefully would be easier to trade. The advantages of the DFT and it's fast computation algorithm the FFT, are that it is extremely fast in calculating the frequencies of the input price series. In addition it can determine frequency peaks for very noisy price series even when the signal amplitude **is less than** the noise amplitude. One of the disadvantages of the FFT is that straight line, parabolic trends and edge effects in the price series can distort the frequency spectrum. In addition, end effects in the price series can distort the frequency spectrum. Another disadvantage of the FFT is that it needs a lot more data than MESA for spectral resolution. However this disadvantage has largely been nullified by the speed of today's computers .

## Comparison of MESA vs DFT Signal/Noise Ratios Frequency Resolution.

Here we will compare the ability of MESA and the DFT to resolve frequencies with various noise levels. We will construct a signal with two periods of 11 bars and 19 bars per cycle, equal amplitudes and separated by a phase difference of 45 degrees. The signal is given by formula (3)

$$Signal(i) = \sin(2*\pi*i/11) + \sin(2*\pi*i/19 + \pi/4) \quad i= 1 \text{ to } n \qquad (3)$$

For noise, the amplitude is defined by the noise standard deviation sigma or $\sigma$ . For our noise computations we will use a uniform distribution, u(i), that will create a random number between +1 and –1. Thus at each point "i" the signal plus noise will be

$$Signal + Noise = signal(i) + u(i) \quad i=1 \text{ to } n$$

Using engineering terms the signal to noise ratio is defined as

$$S/N \text{ in db} = 20*\log10(A/\sigma) \quad (4)$$

Where log10 is the logarithm to the base 10.

Since the amplitude of any frequency of our constructed formula (3) is one, the signal to noise ratio becomes.

$$S/N = 20*\log10(1/\sigma) \text{ db} \qquad (5)$$

We will examine the frequency resolution for both MESA and DFT for signal to noise ratios of +6db ( signal amplitude / $\sigma$ = 2), 0db (signal amplitude/ $\sigma$ = 1), -6db (signal amplitude/$\sigma$ =1/2). For MESA, we will normalize the frequency amplitudes found by equation (2) by dividing all frequency amplitudes found by equation (2) by the highest frequency amplitude found in each 5 to 25 bars per cycle search. This will make the highest frequency power equal to 1 and all others less than one. This normalization procedure will allow us to better compare the frequency resolutions of MESA vs DFT for their power spectrums produce greatly different magnitudes. In addition, for each noise level we will produce 10 trials and average the trials before comparison. We need to produce 10 trials because the noise generator produces random amplitudes that sometimes enhance or minimize certain frequencies so we need to average over 10 trials to minimize random frequency outliers generated by the noise process.

For MESA, we will use 60 data points. From Marple (Ref 11), for noisy data, the optimum number of coefficients is between N/3 and N/2. We will use 20 autoregressive coefficients for our test. We will compute the frequency power at the periods of 5 bars/cycle through 25 bars/cycle (note: a period of 5 bars per cycle is equal to 1/5 cycles per bar).

For DFT, we will use 128 data points. However, in order to find the frequency power at periods 5 through 25 instead of calculating the DFT at equally spaced frequencies of 1/128, we will use a subset of the discrete Fourier transform called the ***Goertzel algorithm*** that is used extensively in tone detection in telephones and cellular phones. The derivation of the Goertzel Algorithm is readily available in the literature ref [8]-[10] and in hundreds of references on the web. The Goertzel algorithm is much faster than the DFT because it searches for one frequency at a time instead of N/2 frequencies all at once. If the number of frequencies searched for is small the Goertzel algorithm can even be faster than the FFT. In addition the Goertzel Algorithm can interpolate between the 1/N frequency divisions so that the frequency search is not limited to just the 1/N spacing of the DFT. While it can detect the frequency within the 1/N spacing it cannot detect more than one frequency within that spacing. For instance if N=20, and we are looking for a frequency between 1/20 and 2/20, the Goertzel can detect the frequency anywhere at or between these two frequencies. However, if there was a second frequency in-between these two frequencies, Goertzel could not find it. If there were two frequencies between the 1/20 and 2/20 values, Goertzel would produce one frequency as a weighted average of the two. This is where MESA has a clear advantage.

The Goertzel Algorithm for any frequency is simple to compute and is given by the time domain difference equation

$$\mathbf{v(n) = 2*cos(2\pi f)*v(n-1) - v(n-2) + price(n) \qquad n=1 \text{ to } N}$$

where N equals total number of prices and v(0) and v(-1) are zero. Price(1) equals first price and Price(N) equals last price.

The amplitude of frequency f is calculated only after the Nth iteration of the above formula and is given by

$$\mathbf{Amp(f)^2 = v^2(N) + v^2(N-1) - 2cos(2\pi f)*v(N)*v(N-1)}$$

## Results

Table 1 presents a comparison the frequency power spectrums of the signal+noise for MESA and FFT(Goertzel) for a S/N ratio of 6 DB(signal amplitude = 2*noise amplitude). As we can see at 6 db, where the signal is two times the noise, both MESA and the Goertzel algorithm(GA) did a good job in finding the periods of 11 and 19 bars per cycle. At the 19 period, MESA had a narrower peak than the GA, while at the 11 period cycle the GA had a narrower peak than MESA.

Table 2 presents a comparison the frequency power spectrums of the signal+noise for MESA and FFT(Goertzel) for a S/N ratio of 0 DB (signal amplitude = noise amplitude). As we can see at 0 db, where the signal strength is equal to the noise strength, both MESA and the Goertzel algorithm were able to identify the periods of 11and 19 bars per cycle. However the GA produced sharper and narrower peaks than MESA making the 11 and 19 periods easier to identify. Also we can see that MESA's cycle discriminating ability is starting to get at lot worse when compared to it's 6 DB power spectrum.

Table 3 presents a comparison the frequency power spectrums of the signal+noise for MESA and Goertzel for a S/N ratio of -6 DB (signal amplitude = ½ noise amplitude). As we can see at -6 db, where the noise strength is twice the signal strength, MESA completely failed to identify the 11 and 19 bar periods. The GA, however, was still able to identify the 11 and 19 bar periods with approximately the same resolution as before.

## Summary

We have examined the frequency analysis techniques of the Maximum Entropy Spectral Analysis(MESA) and the Goertzel Algorithm (GA) with various noise levels added to a simple sinusoidal functions. While MESA can do an excellent job of identifying frequencies within data when the noise strength is half the signal strength, MESA's abilities start to deteriorate fast when the when the noise strength becomes greater than the signal strength. With very noisy data where the noise strength is greater than the signal strength, as Table 3 shows, only the Goertzel Algorithm can successfully identify the frequencies present.

In our next follow-up article we use this working paper's conclusions and will present a system that uses the Goertzel –DFT algorithm to find the best frequencies in the price series. We will use the amplitude and phases of these frequencies to produce an estimate of the next bars price. The system will walk forward producing the next bars price and follow a next bars price curve to generate it's buy and sell signals similar to the End Point Fast Fourier Transform papers on our web site at http://meyersanalytics.com/articles.htm .

**References:**


1. Burg, J. P., 'Maximum Entropy Spectral Analysis", Ph.D. dissertation, Stanford University, Stanford, CA. May 1975.
2. Kay, Steven M., "Modern Spectral Estimation", Prentice Hall, 1988
3. Marple, Lawrence S. Jr., "Digital Spectral Analysis With Applications", Prentice Hall, 1987
4. Press, William H., et al, "Numerical Receipts in C++: the Art of Scientific Computing", Cambridge Press, 2002.
5. Oppenheim, A, Schafer, R. and Buck, J., "Discrete Time Signal Processing", Prentice Hall, 1996, pp663-634
6. Proakis, J. and Manolakis, D. "Digital Signal Processing-Principles, Algorithms and Applications", Prentice Hall, 1996., pp480-481
7. Goertzel, G., "An Algorithm for he evaluation of finite trigonometric series" American Math Month, Vol 65, 1958 pp34-35.


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# Table 1 MESA vs GA  Frequency Amplitude Strength S/N Ratio= 6DB

| Frequency | Period | MESA Average 10 Trials | GA Average 10 Trials |
|---|---|---|---|
| 0.0400 | 25 | **0.01** | **0.04** |
| 0.0417 | 24 | **0.01** | **0.05** |
| 0.0435 | 23 | **0.02** | **0.03** |
| 0.0455 | 22 | **0.04** | **0.02** |
| 0.0476 | 21 | **0.09** | **0.23** |
| 0.0500 | 20 | **0.38** | **0.71** |
| 0.0526 | 19 | **0.60** | **0.98** |
| 0.0556 | 18 | **0.23** | **0.56** |
| 0.0588 | 17 | **0.05** | **0.05** |
| 0.0625 | 16 | **0.02** | **0.04** |
| 0.0667 | 15 | **0.01** | **0.01** |
| 0.0714 | 14 | **0.01** | **0.03** |
| 0.0769 | 13 | **0.02** | **0.02** |
| 0.0833 | 12 | **0.04** | **0.01** |
| 0.0909 | 11 | **0.72** | **0.94** |
| 0.1000 | 10 | **0.01** | **0.03** |
| 0.1111 | 9 | **0.00** | **0.03** |
| 0.1250 | 8 | **0.00** | **0.02** |
| 0.1429 | 7 | **0.00** | **0.01** |
| 0.1667 | 6 | **0.01** | **0.01** |
| 0.2000 | 5 | **0.00** | **0.01** |

# Table 2 MESA vs GA  Frequency Amplitude Strength S/N Ratio= 0 DB

| Frequency | Period | MESA Average 10 Trials | GA Average 10 Trials |
|---|---|---|---|
| 0.0400 | 25 | **0.13** | **0.05** |
| 0.0417 | 24 | **0.16** | **0.07** |
| 0.0435 | 23 | **0.16** | **0.04** |
| 0.0455 | 22 | **0.17** | **0.02** |
| 0.0476 | 21 | **0.27** | **0.20** |
| 0.0500 | 20 | **0.57** | **0.65** |
| 0.0526 | 19 | **0.72** | **0.91** |
| 0.0556 | 18 | **0.34** | **0.51** |
| 0.0588 | 17 | **0.16** | **0.05** |
| 0.0625 | 16 | **0.09** | **0.05** |
| 0.0667 | 15 | **0.06** | **0.02** |
| 0.0714 | 14 | **0.05** | **0.05** |
| 0.0769 | 13 | **0.05** | **0.03** |
| 0.0833 | 12 | **0.11** | **0.04** |
| 0.0909 | 11 | **0.58** | **0.80** |
| 0.1000 | 10 | **0.14** | **0.04** |
| 0.1111 | 9 | **0.02** | **0.05** |
| 0.1250 | 8 | **0.01** | **0.03** |
| 0.1429 | 7 | **0.01** | **0.04** |
| 0.1667 | 6 | **0.03** | **0.04** |
| 0.2000 | 5 | **0.02** | **0.02** |

## Table 3 MESA vs GA Frequency Amplitude Strength S/N Ratio= -6 DB

| Frequency | Period | MESA Average 10 Trials | GA Average 10 Trials |
|---|---|---|---|
| 0.0400 | 25 | **0.48** | **0.20** |
| 0.0417 | 24 | **0.53** | **0.21** |
| 0.0435 | 23 | **0.59** | **0.17** |
| 0.0455 | 22 | **0.63** | **0.12** |
| 0.0476 | 21 | **0.64** | **0.23** |
| 0.0500 | 20 | **0.59** | **0.58** |
| 0.0526 | 19 | **0.46** | **0.82** |
| 0.0556 | 18 | **0.34** | **0.53** |
| 0.0588 | 17 | **0.28** | **0.16** |
| 0.0625 | 16 | **0.24** | **0.11** |
| 0.0667 | 15 | **0.22** | **0.09** |
| 0.0714 | 14 | **0.24** | **0.23** |
| 0.0769 | 13 | **0.31** | **0.25** |
| 0.0833 | 12 | **0.58** | **0.10** |
| 0.0909 | 11 | **0.57** | **0.76** |
| 0.1000 | 10 | **0.22** | **0.14** |
| 0.1111 | 9 | **0.11** | **0.10** |
| 0.1250 | 8 | **0.12** | **0.08** |
| 0.1429 | 7 | **0.14** | **0.16** |
| 0.1667 | 6 | **0.12** | **0.16** |
| 0.2000 | 5 | **0.11** | **0.19** |